Calculation of combined diffusive and convective mass transfer

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ABSTRACT

The clearance of a dialyzer is calculated under the most general conditions, allowing not only for a mixed diffusive and convective mass transfer, but also for a variation along the membrane of the local ultrafiltration, the membrane permeability and the sieving coefficient. The study is then carried on for the case in which these are all constant, to reach a relatively simple expression for the influence of a low ultrafiltration rate on the clearance. In this study, the permeabilities of the boundary layers on both sides are treated as included in the (equivalent) membrane. In an appendix, the stacking of membranes is studied, giving a general law for the calculation of overall permeabilities of a stack of individual membranes, regarded as one (equivalent) membrane (such as a physical membrane with two boundary layers). Permeability data for boundary layers are quoted from earlier works. In other appendices, the variation of the local ultrafiltration along the dialysis path is studied, as well as its effect on the effective permeability of the membrane.

1. General calculation of the clearance of a dialyzer

A segment of a dialyzer in countercurrent operation is shown in Fig. 1. The effects of boundary layers are included in the membrane. We allow for a variation along the axial coordinate $x$ of both ultrafiltration and membrane permeability data.

In Fig. 1, the following notations are used:

- $k$ = permeability factor
- $L$ = active length
- $A$ = total membrane surface area
- $S$ = sieving coefficient
- $C_{b}$ = blood concentration of solute
- $C_{d}$ = dialysate concentration of solute
- $Q_{b}$ = blood flow
- $Q_{d}$ = dialysate flow
- $q_{u}$ = axial density of the ultrafiltration flow
- $x$ = axial coordinate from the blood entrance end.

Below, the clearance is denoted by $Q_{C}$ and the total ultrafiltration by $Q_{U}$. Furthermore, $K = kA_{L}$ and $C_{bL} = C_{b}$ (L). Other notations are introduced in the text and all are listed at the end of it.

Here, $k$, $S$ and $q_{u}$ may be functions of $x$. Accordingly, $C_{b}$, $C_{d}$, $Q_{b}$, $Q_{d}$ and $K$ are functions of $x$.

Mass balance yields

$$
\begin{align*}
- \frac{dQ_{b}}{dx} + Q_{U}C_{b} &= (K + S q_{u}) C_{b} - K C_{d}, \\
- \frac{dQ_{d}}{dx} + Q_{U}C_{d} &= (K + S q_{u}) C_{b} - K C_{d}.
\end{align*}
$$

Rearranged:

$$
\begin{align*}
\frac{dQ_{b}}{dx} &= \left( K - (1 - S) q_{u} \right) C_{b} - K C_{d}, \\
\frac{dQ_{d}}{dx} &= \left( K + S q_{u} \right) C_{b} - (K + q_{u}) C_{d}.
\end{align*}
$$

Furthermore:

$$
\begin{align*}
\frac{dQ_{b}}{dx} &= - q_{u}, \quad Q_{b} = Q_{b0} - \frac{x}{S} q_{u} dx, \\
\frac{dQ_{d}}{dx} &= - q_{u}, \quad Q_{d} = Q_{d0} - \frac{x}{S} q_{u} dx, \\
Q_{dl} + Q_{u} &= \frac{x}{S} q_{u} dx.
\end{align*}
$$

wherein $Q_{b0} = Q_{b}(0)$, $Q_{d0} = Q_{d}(0)$ and $Q_{dl} = Q_{d}(L)$.

A mass balance over the section from $x$ to $L$ gives, if $C_{d}(L) = 0$ (the case $C_{d}(L) \neq 0$ is easily handled through superposition of concentrations in the usual, known manner),

$$
Q_{b}C_{b} - Q_{b}C_{bL} = Q_{d}C_{d},
$$

or

$$
C_{d} = \frac{Q_{b}C_{b} - Q_{b}C_{bL}}{Q_{d}}.
$$

With (3), one then finds

$$
- \frac{dQ_{b}}{dx} = \left[ \left( 1 - \frac{Q_{b}}{Q_{d}} \right) \left( 1 - S q_{u} \right) \right] C_{b} - \frac{K_{b} C_{bL}}{Q_{d}} C_{bL}.
$$

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One may first solve the corresponding homogeneous equation:

$$- Q_b \frac{dC_b}{dx} = \left[ K(1 - \frac{Q_b}{Q_d}) - (1 - S) \frac{Q_u}{Q_b} \right] C_b. \quad (10)$$

Introducing

$$f(x) = K(1 - \frac{1}{Q_b} - \frac{1}{Q_d}) - (1 - S) \frac{Q_u}{Q_b}, \quad (11)$$

one finds

$$C_b = \beta e^{- \int f(n) \, dn}, \quad (12)$$

where \( \beta \) is a constant.

Applying the method of «varying the constant», (9) can now be solved. Introducing

$$g(x) = \frac{KQ_bL}{Q_b Q_d}, \quad (13)$$

(9) can be written as

$$\frac{dC_b}{dx} = - f(x) C_b - g(x) C_b L, \quad (14)$$

with the solution

$$C_b = e^{\int f(n) \, dn} \left[ C_0 - C_b L \int_{0}^{L} g(\xi) e^{\int_{0}^{\xi} f(n) \, dn} \right], \quad (15)$$

Herein, \( C_0 \) is a constant, which amounts to the blood concentration of the solute at \( x = 0 \), i.e., \( C_0 = C_b(0) \). Herewith, \( C_b \) is found from equation (7).

For calculating the clearance, one needs the outlet concentration of the solute in the blood, \( C_{bL} \), which is found through putting \( x = L \) in (15):

$$C_{bL} = e^{\int_{0}^{L} f(n) \, dn} \left[ C_0 - C_{DL} L \int_{0}^{L} g(\xi) e^{\int_{0}^{\xi} f(n) \, dn} \right] = C_D. \quad (16)$$

By definition, the clearance for zero inlet concentration in the dialysate is

$$Q_c = Q_{bo} - Q_{DL} \frac{C_{bL}}{C_D}, \quad (17)$$

from which, with (16) (after rearrangement):

$$Q_c = Q_{bo} \frac{L}{\int_{0}^{L} \left[ f(\xi) + g(\xi) \right] e^{\int_{0}^{\xi} f(n) \, dn} \, d\xi}. \quad (18)$$

In most cases, \( q_u \) is an almost linear function of \( x \), and pressures in blood and dialysate, resp., usually fall almost linearly in the flow direction (cf. Appendix 1). One may therefore put

$$q_u = m - nx, \quad (19)$$

wherein \( m \) and \( n \) are constants. One then finds, with (5) and (6),

$$Q_b = Q_{bo} - mx + \frac{nx^2}{2}, \quad (20)$$

$$Q_d = Q_{DL} + Q_u - mx + \frac{nx^2}{2}, \quad (21)$$

referred to inlet flows \( Q_{bo} \) and \( Q_{DL} \), resp.

Where applicable, the expressions (19) - (21) are, therefore, to be inserted into \( f(x) \) and \( g(x) \). Furthermore, one generally has (whatever the course of \( q_u \) with \( x \)):

$$Q_{DL} = Q_{bo} - Q_u \text{ and } Q_{DL} = Q_{DL} + Q_u.$$

Under the condition (A.19), which holds for most dialyzers, one finds from Appendix 1:

$$q_u = \frac{Q_u}{L} + \frac{(L - x) U}{L} \left( \Delta p_b + \Delta p_d \right), \quad (22)$$

wherein \( \Delta p_b \) and \( \Delta p_d \) are the pressure drops in the blood and dialysate pathways of the membrane section of the device, resp., here taken as approximately constant for various \( Q_u \). This follows from (A.16) and the mean transmembrane pressure

$$P_{tm} = P_{tmo} - \frac{1}{2} (\Delta p_b + \Delta p_d), \quad (23)$$

wherein \( P_{tmo} \) is the transmembrane pressure at \( x = 0 \), combined with (A.21) and (A.22) in Appendix 1. \( U \) is the ultrafiltration coefficient of the membrane:

$$U = \frac{Q_u}{P_{tm}}, \quad (24)$$

which in most cases can be treated as a constant (in reality, it varies somewhat with \( P_{tm} \); for a hollow fiber dialyzer by extremely small amounts and somewhat more for compliant dialyzers).

The contribution from the second term in (22) is in the order of \( \pm 40 \) mHg, or \( \pm 0.7 \) mmHg for typical dialyzers. Integrated over the membrane from \( x = 0 \) to \( x = L \), this second term becomes zero. Therefore, this term may be neglected in comparison with a \( Q_b \) of 200 ml/min and a \( Q_d \) of 500 ml/min, which flow values constitute the standard operating condition for dialyzer evaluation. One
may therefore write

$$q_u = \frac{Q_u}{L},$$

(25)

which renders \( q_u \) constant and considerably simplifies

the application of (18).

For the application in more general cases, it is advantageous to first calculate

$$\frac{C_0}{C_{bl}} \int_0^L \left[ f(\xi) + g(\xi) \right] d\xi,$$

(26)

which follows from (16), and then calculate \( Q_u \) from (17). In the case of a constant \( q_u \), one gets the following:

$$Q_b = \frac{Q_{bo} - q_u}{q_u},$$

(27)

$$Q_d = \frac{Q_{do} - q_u}{q_u},$$

(28)

$$f(\xi) = \frac{K - (1-S)q_u}{Q_{bo} - q_u} - \frac{K}{Q_{do} - q_u},$$

(29)

$$g(\xi) = \frac{K(\frac{Q_{bo} - Q_u}{Q_{do} - Q_u})}{Q_{bo} - Q_{do}} \left( \frac{1}{Q_{bo} - q_u} - \frac{1}{Q_{do} - q_u} \right).$$

(30)

Further limiting to the case of constant \( K \) and \( S \), one finds, using \( KL = kA \) and \( q_uL = Q_u \),

$$\frac{C_0}{C_{bl}} = \frac{(1 - \frac{Q_u}{Q_{do}})}{\frac{Q_u}{Q_{bo}} - 1 + S},$$

(31)

wherein

$$\frac{\Phi}{\Phi_{bo}} = 1 - \frac{Q_u}{Q_{bo}}.$$  

(33)

This integral cannot be generally solved in expressions of elementary functions. For small \( Q_u \), in relation to \( Q_{bo} \) and \( Q_{do} \), one may, however, approximate the integral in (31) as

$$\int_{0}^{L} \left[ \frac{kA + S Q_u}{Q_{bo} - Q_{do}} - \frac{k A - Q_u}{Q_{do}} \right] \frac{d\xi}{L} = \frac{Q_{do} - Q_u}{Q_{bo} - Q_{do}} \left( 1 + \frac{kA(Q_{bo} - Q_u)}{Q_{bo} + Q_{do}} \right) \frac{Q_{bo} - Q_{do}}{Q_{bo} + Q_{do}}.$$  

(34)

and, similarly approximating the first term in (31), one finds

$$\frac{C_0}{C_{bl}} \approx e \frac{Q_u}{Q_{bo}} \left( 1 + \frac{kA(Q_{bo} - Q_u)}{Q_{bo} + Q_{do}} \right) \frac{Q_{bo} - Q_{do}}{Q_{bo} + Q_{do}}.$$  

(35)

These approximations become exact as \( Q_u \to 0 \).

Developing into a series in \( Q_u \) and keeping only first-order terms, one finds, for \( Q_u << Q_{bo}, Q_{dq} \),

$$\frac{C_0}{C_{bl}} \approx \left( \frac{Q_{do} - Q_u}{Q_{bo} - Q_{do}} \right) e \left( 1 + \frac{kA(Q_{bo} - Q_u)}{Q_{bo} + Q_{do}} \right) \frac{Q_{bo} - Q_{do}}{Q_{bo} + Q_{do}}.$$  

(36)

The same expression is found through differentiation of (35).

With (36), the variation of \( C_{bl} \) with small values of \( Q_u \) can be calculated. One herein has to consider the variation of \( kA \) with \( Q_u \), which is discussed in Appendix 2. From (A.28), one approximately has

$$kA = \frac{2(k_o A)^2}{2 k_o A + Q_u S},$$

(37)

wherein \( k_o \) is the value of \( k \) at \( Q_u = 0 \). (37) is valid for a
flat membrane arrangement with the surface area A, or for a hollow-fiber dialyzer with the internal surface area A. However, for a hollow-fiber dialyzer, the surface area A in (36) is the actual surface area, which is the logarithmic mean value of the internal and external surface areas (cf. [3] and Appendix 2). Accordingly, a factor expressing surface relations has to be entered in (37) for a hollow-fiber dialyzer — cf. Appendix 2.

The clearance \( Q_{co} \) at \( Q_u = 0 \) is found from (17) and (35) (being exact at \( Q_u = 0 \)):

\[
Q_{co} = Q_{bo} Q_{do} \left(1 - e^{-k_A Q_{do}/Q_{bo}} \right)
\]  

(38)

from which one finds

\[
k_A = \frac{Q_{do}/Q_{bo}}{ln \left(\frac{Q_{bo}/Q_{do} - Q_{co}}{Q_{bo}/Q_{do} - Q_{co}}\right)}
\]

(39)

Expanding (37) and the exponential function of \( k_A \) in series, one then finds for small values of \( Q_u \)

\[
\Delta \left(\frac{C_o}{C_{BL}}\right) = \frac{C_o}{C_{BL}} - \frac{C_{bo}}{C_{BL}(O)} - \frac{Q_u}{Q_{bo}} \left(\frac{Q_{do} - Q_{bo}}{Q_{do} - Q_{co}}\right) \left(\frac{Q_{bo}^2}{Q_{do}} - 1 + S\right) + \frac{Q_u Q_{co} (S + Q_{do})}{Q_{do} (Q_{bo} - Q_{co})} - \frac{Q_u S Q_{do} - Q_{bo}}{Q_{bo} - Q_{co}}
\]

(40)

wherein \( C_{BL}(O) \) is \( C_{BL} \) at \( Q_u = 0 \).

The standard operating condition for a dialyzer is at \( Q_{bo} = 200 \text{ ml/min} \) and \( Q_{do} = 500 \text{ ml/min} \) (rather than \( Q_{DL} = 500 \text{ ml/min} \), since the suction which creates ultrafiltration is usually generated by means of a controlled dialysate pump placed at the outlet side of the dialyzer).

Under this condition, one finds

\[
\Delta \left(\frac{C_o}{C_{BL}}\right) = \frac{Q_u}{300} \left[\frac{500 - Q_{co}}{200 - Q_{co}} - (S - 0.84) + 1\right]
\]

\[
- \frac{Q_u Q_{co} (S + 0.4)}{500(200 - Q_{co}) \ln \left(\frac{0.4 - 500 - Q_{co}}{200 - Q_{co}}\right)}
\]

(41)

if \( Q_{co} \) and \( Q_u \) are inserted in ml/min.

For small values of \( Q_u \), the clearance according to (17) is

\[
Q_c = Q_{co} + Q_u \left[\frac{C_{BL}(O)}{C_o} \right] \frac{C_{BL}(O)}{C_o} \Delta \left(\frac{C_o}{C_{BL}}\right)
\]

and, furthermore,

\[
\frac{C_{BL}(O)}{C_o} = 1 - \frac{Q_{co}}{Q_{bo}}
\]

(43)

One therefore finds

\[
Q_c = Q_{co} + Q_u \left[1 - \frac{Q_{co}}{Q_{bo}}\right] + \frac{Q_{bo} (1 - \frac{Q_{co}}{Q_{bo}})^2}{Q_{bo}} \Delta \left(\frac{C_o}{C_{BL}}\right)
\]

(44)

wherein \( \Delta \left(\frac{C_o}{C_{BL}}\right) \) is found from (40) or (41). With this, one can now calculate the variation of \( Q_c \) with \( Q_u \) for \( Q_u \ll Q_{bo}, Q_{do} \). For this purpose, one may first define

\[
\Delta Q_{c1} = \frac{Q_u (1 - \frac{Q_{co}}{Q_{bo}})}{Q_{bo}}
\]

(45)

and

\[
\Delta Q_{c2} = \frac{Q_u (1 - \frac{Q_{co}}{Q_{bo}})^2}{Q_{bo}} \Delta \left(\frac{C_o}{C_{BL}}\right)
\]

(46)

\( \Delta Q_{c1} \) is the common estimate for the alteration of \( Q_c \) with \( Q_u \), since experimental studies show that \( C_{BL} \) remains practically unaltered under ultrafiltration. Nevertheless, \( \Delta Q_{c2} \) is found to give a significant contribution, as will be seen in the following.

With the relations so found, one can calculate \( \Delta C_{BL} = C_{BL}(O) - C_{BL} \), as well as \( \Delta Q_{c1} \) and \( \Delta Q_{c2} \). The results are shown for \( S = 0.5 \) and \( S = 1 \) under the standard operating condition in Figs. 2 and 3. In Fig. 2, \( \Delta C_{BL} \) per unit ultrafiltration (ml/min) is given in % of \( C_{BL} \). In Fig. 3, the negative sign of \( \Delta Q_{c2} \) is to be noted. It is seen that
$\Delta Q_{bL}/Q_b$ falls in the range of 1-4% of $C_{bL}$, but still $\Delta Q_{C}$ falls between $-29\%$ and $-66\%$ of $\Delta Q_{C1}$ in practical cases. This can be understood from (45) and (46), in which the factor $Q_u$ in $\Delta Q_{C1}$ is much smaller than the factor $Q_{bo}$ in $\Delta Q_{C2}$. Therefore, correction of clearance according to (45) alone is obviously disputable. (Ranges given for values of S between 0.5 and 1 — smaller values are rarely actual for typical solutes used in dialyzer evaluation).

It is of interest to compare this with measured values, even though measurements usually are to a relatively high extent influenced by limitations in the accuracy of the chemical analysis, since one has to deal with very small changes in the output concentration (vide supra). In spite of a relatively high scatter in individual figures at various $Q_u$ even for the same dialyzer, values of $\Delta Q_{C}/\Delta Q_{U}$ estimated by means of linear regression are given in Table 1, as determined at the HRC in Salt Lake City in 1979 for three types of dialyzers. It is seen that (41) and (44) fit these values exactly with plausible values of S for the vitamin B12 clearances in all cases, whereas the urea value is fitted only for «Hemoflow C 0.8», but then exactly at the actual value of S for urea convection through Cuprophan, i.e., S = 0.98 (1).

The use of (45) alone leads to much too high values for vitamin B12, but roughly acceptable values for urea. The measured urea values in Table 1 should, however, be taken with caution, since these are subject not only to scatter to a higher degree than the values for vitamin B12, but also to the influence of a non-ideal distribution of the dialysate flow in the hollow-fiber bundle (especially pronounced in the «RDi» dialyzer) — the latter influence is considerably lower for vitamin B12. The lack of fit for certain urea values is therefore likely to be caused by such other effects on the measured values. It would, of course be of interest if an accurate study would be performed, requiring a high number of measurements at several ultrafiltration rates for a statistical reduction of the effect of scatter. Measurements for urea here require much higher accuracy in the chemical analysis, since $C_{bL}$ is 5 or 6 times higher for vitamin B12 than it is for urea. This requirement is hampered by the fact that both the Jaffé and the enzymatic methods of analysis are of comparatively low accuracy. Therefore, other methods should be used.

It may be remarked that if the membrane permeability is taken as constant, i.e., if $k_o$ is used in (36) instead of k according to (37) [which corresponds to dropping the last term in (40) and (41)], a poorer fit is reached (except for the dubious urea values) in Table 1.

2. A special case

If $Q_{bo} = Q_{bo}$, the integral in (31) can be expressed in elementary functions. One finds

$$C_{bl} = (1 - \frac{kA}{Q_uS} (1 - \frac{Q_u}{Q_{bo}})) - \frac{kA}{Q_uS} (1 - \frac{Q_u}{Q_{bo}})$$

(47)

If S = 0. The case S = 0 gives (2)

$$C_{bl} = (1 - \frac{Q_u}{Q_{bo}}) [1 - \frac{kA}{Q_u} (1 - \frac{Q_u}{Q_{bo}})]$$

(48)

(1) here the surface-relation factor for hollow fibers, mentioned in the text below (37), has been neglected, which is a reasonable approximation.

(2) this is purely theoretical, for the mathematical interest only, since a physical membrane with S = 0 could hardly have a diffusive permeability.

*for the same $C_0$

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<td>RDi</td>
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</tbody>
</table>

$Q_b$ is given in ml/min at $Q_{bo} = 200$ ml/min and $Q_{bo} = 500$ ml/min
Dialyzer clearance

The other extreme case $S = 1$ gives

$$
\frac{C_{0}}{C_{BL}} = 1 + \frac{kA}{Q_{bo}} \tag{49}
$$

independent of $Q_{bo}$.

3. Calculations of the membrane permeability

Above, the effects of boundary layers were included in the (equivalent) membrane. For practical applications of the above relations, one therefore needs estimates of the permeabilities of those layers, as well as relations for combined (stacked) membranes (since one may regard the physical membrane and the two boundary layers as three membranes stacked one upon the other).

In Appendix 3, a study of combined membranes is undertaken. From this, one can calculate the effective or equivalent overall permeability of any number of membranes stacked together, regarded as one single membrane.

In the actual case, the middle membrane is the physical one and its permeability data are therefore given. The other two "membranes" are equivalent to the effects of the two boundary layers on the blood side and on the dialysate side, respectively. For calculation of the overall permeability data, one needs expressions for the latter two "membranes". In [2] a calculation is set up for the dialysate side of hollow fibers (pp. 62-63), which is not carried through to a final expression because of lengthiness. However, the result of such a calculation is found to be

$$
k_D = \frac{72 D_d}{r_2^2 W} \tag{50},
$$

$$
V = \left(3 - 4t^2 + t^4 + 4 \ln t\right)^2, \tag{51}
$$

$$
W = -719 + 1680 t^2 - 1296 t^4 + 368 t^6 - 33 t^8 - 120 (19 - 24 t^2 + 6 t^4) \ln t - 288 (9 - 4 t^2)^2 \ln t - 1152 t^3, \tag{52}
$$

wherein $k_D$ is the permeability factor for the dialysate boundary layer and

$$
t = r_2 \sqrt{\frac{\ln N}{A_D}}. \tag{53}
$$

Here $r_2$ is the external radius of the wet (swollen) fiber and $D_d$ is the diffusion constant in the dialysate. $N$ is the total number of fibers in the bundle and $A_D$ is the total cross-sectional area of the bundle (including fiber interspaces).

The blood-side boundary layer permeability is calculated by Babb et al. [4] to be

$$
k_B = \frac{D_b}{0.25 h} \tag{54}
$$

for a flat membrane dialyzer, wherein $h$ is the full blood-channel height (from membrane to membrane) and $D_b$ is the diffusion constant in the blood. Klein et al. [5] have adopted the same relation for a hollow-fiber dialyzer, simply putting $h = 2r_1$, where $r_1$ is the internal radius of the wet fiber. The theory in [2] allows for a more exact calculation of this permeability for hollow fibers. As a result,

$$
k_B = \frac{D_b}{2\pi r_1}, \tag{55}
$$

wherein

$$
\alpha = 2 \left(\frac{1}{\beta^2} - \frac{1}{4w}\right), \tag{56}
$$

Herein, $p_1$ is the first positive root of

$$
- p^{10}(0.421880 + w 0.0926930)10^{-7} + p^{6}(0.566862 + w 0.145445)10^{-5} - p^{2}(0.450304 + w 0.144043)10^{-3} + p^{4}(0.0182292 + w 0.00792101) - p^{2}(0.25 + w 0.1875) + w = 0 \tag{57}
$$

(which is (83) of [2] somewhat corrected and extended). $w$ is here a parameter:

$$
w = \frac{k' r_1}{D_b}, \tag{58}
$$

wherein $k'$ is the total (combined) permeability of the physical membrane and the dialysate boundary layer (cf. Appendix 3).

The factor $\alpha$ in (55) is shown as a function of $w$ in fig. 4. It is found that $0.229 < \alpha < 0.274$, so that the value 0.25 of [5] is a reasonable approximation.

For a flat membrane dialyzer, [4] states that the permeability of the dialysate boundary layer would be negligible at high values of $Q_d$, since then $Q_c$ levels off to become almost constant, taken as a function of $Q_d$. However, as is seen from (58), this is no proof for negligibility of the dialysate boundary layer, since $Q_c$ levels...
off in any case, whether there is a significant boundary layer resistance, or not. Furthermore, there is no reason to expect that this permeability would be flow dependent, since the permeability of the blood side boundary layer is not! Instead, for reasons of symmetry, both boundary layer permeabilities should depend on geometries (or: relative velocity gradients at the membrane), and not on flow values. By analogy to (54), the dialysate boundary layer should therefore have a permeability

$$k_D = \frac{D_d}{\lambda_d}$$  \hspace{1cm} (59)

in a flat membrane dialyzer, wherein $d$ is the height of the dialysate channel. $\lambda$ is a coefficient which differs from the value in (54) even at laminar dialysate flows, since the dialysate channel has a membrane on one side only, and a solid wall on the other, making for an unsymmetrical concentration distribution across the channel. In many flat membrane dialyzers, the dialysate furthermore flows in a complicated manner inside a membrane support structure (such as a mesh structure or a system of pyramidal protrusions from the solid wall), making for a mixing in the dialysate through multiple local turbulence. An estimate of $\lambda$ is therefore very difficult to find and no useful literature source is known to the author.

Estimate of boundary layer permeabilities here given are for overall effects, not considering their variations with $x$. They furthermore apply to the case $q_u = 0$ and therefore correspond to $k_D$ in (37) and Appendices 2 and 3. As a first approximation one may put $S = 1$ for boundary layers, but more accurate values should be derived from studies of mass transport in fluid channels with flow components perpendicular to the $x$-axis — no such study is known to the author.

In Appendix 3, the radial stacking of round, tubular membranes (such as hollow fibers and associated boundary layers) is also discussed. It is found that one has to refer the individual membrane permeabilities to a specific radius, such as the inner radius of the fiber wall. $k_D$ above is already referred to this inner wall radius $r_1$ in the hollow-fiber case. $k_D$ above, is, however, referred to the outer wall radius $r_2$ in the hollow-fiber case and can be referred to $r_1$ through multiplying by $r_2/r_1$ (cf. Appendix 3).

### APPENDIX 1

**Study of the ultrafiltration flow density**

$q_u(x)$ is the axial ultrafiltration flow density and $q_u(x)A$ the ultrafiltration flow density per unit area, assumed to be uniform, i.e., constant, in a direction perpendicular to $x$ across the membrane.

In most cases, the pressures in the blood and dialysate paths drop linearly in the flow direction at zero ultrafiltration. Correspondingly, we can define overall (R) and differential (r) flow resistances:

$$R_b = \frac{\Delta p_b}{D_b}$$  \hspace{1cm} (A.1)

$$R_d = \frac{\Delta p_d}{D_d}$$  \hspace{1cm} (A.2)

### TABLE II - TABLE OF VALUES USED FOR FIGS. 2 AND 3

<table>
<thead>
<tr>
<th>$Q_{bo}$ m/min</th>
<th>$\Delta Q_{bo}$</th>
<th>$\Delta Q_{bo}Q_{bo} \times 1000$ min/ml</th>
<th>$\Delta Q_{bo}Q_{bo}$</th>
<th>$\Delta Q_{bo}Q_{bo}$</th>
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<td>$S = 1$</td>
<td>$S = 0.1$</td>
<td>$S = 0.1$</td>
<td>$S = 0.5$</td>
<td>$S = 0.5$</td>
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</table>
wherein $\Delta p_b^O$ is the pressure drop in the blood path over the membrane and $\Delta q_d^O$ is the corresponding drop on the dialysate side, both at $Q_u = 0$. Blood and dialysate flows at $Q_u = 0$ are denoted by $Q_b^O$ and $Q_d^O$, resp. Obviously, from the previous analysis, $\Delta q_d^O = Q_d^O$ or the entrance blood flow, and $Q_d^O = Q_d$ under usual operating conditions [under special conditions one could have $Q_d^O = Q_d$ (L)].

In the general case, one then has a transmembrane pressure

$$P_{tm}(x) = P_{tm}^O - \int_0^x Q_b^O(\xi) \, d\xi - \int_0^x Q_d^O(\xi) \, d\xi,$$

(A.3)

where $P_{tm} = P_{tm}(O)$. Hence, from (5) and (6):

$$\frac{dP_{tm}}{dx} = -\int_0^x \frac{Q_b^O(\eta)}{\eta} \, d\eta + \int_0^x \frac{Q_d^O(\eta)}{\eta} \, d\eta,$$

(A.4)

$$\frac{d^2P_{tm}}{dx^2} = \left( \frac{Q_b^O + Q_d^O}{\eta} \right) \frac{dx}{\eta},$$

(A.5)

$$\frac{dx}{\eta} = \left( \frac{Q_b^O + Q_d^O}{\eta} \right) \frac{dx}{\eta},$$

(A.6)

$q_u$ is proportional to $P_{tm}$, and one can put

$$q_u(x) = u^2 P_{tm}(x),$$

(A.7)

wherein $u$ is a constant. Hence, from (A.6)

$$\frac{d^2q_u}{dx^2} = u^2 \left( \frac{Q_b^O + Q_d^O}{\eta} \right) q_u,$$

(A.8)

with the solution

$$q_u = ae \sqrt{\frac{a}{b}} + be \sqrt{-ae \sqrt{\frac{a}{b}} + \sqrt{\frac{a}{b}}},$$

(A.9)

wherein $a$ and $b$ are constants.

At $x = 0$, one then finds, from (A.4) and (A.5),

$$q_u(O) = u^2 P_{tm} = a + b,$$

(A.10)

$$\left\{ \begin{array}{l}
\frac{dq_u}{dx} = -u^2 \left( \frac{Q_b^O + Q_d^O}{\eta} \right) \frac{dx}{\eta}, \\
2a = u^2 P_{tm} - \frac{Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}}}, \\
2b = u^2 P_{tm} + \frac{Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}}},
\end{array} \right.$$}

(A.11)

from which

$$\left\{ \begin{array}{l}
2a = u^2 P_{tm} - \frac{Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}}}, \\
2b = u^2 P_{tm} + \frac{Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}}},
\end{array} \right.$$}

(A.12)

$$\left\{ \begin{array}{l}
2a = u^2 P_{tm} - \frac{Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}}}, \\
2b = u^2 P_{tm} + \frac{Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}}},
\end{array} \right.$$}

(A.13)

It follows that

$$Q_u = \int_0^L \frac{1}{u \sqrt{a + b}} \left( \frac{L}{u \sqrt{a + b}} \right) \left( \frac{L}{u \sqrt{a + b}} \right) = \frac{1}{u \sqrt{a + b}} \left( \frac{L}{u \sqrt{a + b}} \right),$$

(A.14)

and, since the mean transmembrane pressure is

$$P_{tm} = \frac{1}{L} \int_0^L \frac{dx}{u \sqrt{a + b}} = \frac{1}{L} \int_0^L \frac{u_q \, dx}{u \sqrt{a + b}},$$

(A.15)

one finds

$$Q_u = u^2 L P_{tm} = u^2 L P_{tm},$$

(A.16)

or

$$u = \sqrt{\frac{L}{U}},$$

(A.17)

wherein $U$ is the ultrafiltration coefficient of the membrane.

From (A.9), (A.12) and (A.13), one then can write $q_u$ as

$$q_u = u^2 P_{tm} \cosh \left( xu \sqrt{\frac{a}{b}} + \frac{u}{b} \right) - \frac{u^2 Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}} + \frac{u}{b}},$$

(A.18)

From this, it follows:

1. small values of $Q_u$ in relation to $Q_b$ and $Q_d$, are possible only if

$$Lu \sqrt{\frac{a}{b}} + \frac{u}{b} << 1,$$

(A.19)

(assuming that the operating conditions are such that $q_u \geq 0$ for all $x$, since where $q_u < 0$ dialysate could otherwise be infused into the blood in case of a membrane leak); otherwise, pressure drops alone make for a high ultrafiltration.

2. under this condition, $q_u$ is an approximately linear function of $x$ at all ultrafiltration values.

An evaluation of existing membranes at typical dialyzer pressure drops indicates that the condition (A.19) is nearly always fulfilled (except, e.g., for long dialyzers with the Membrana HDF Cuprophan membrane at more or less original permeability, i.e., without substantial permeability losses in the dialyzer manufacturing procedure). Therefore, one can in most cases assume a nearly linear function $q_u(x)$ for dialyzers, but usually not for hemofilters and plasmapheresis filters.

Cases in which (A.19) does not hold, are such in which the pressure drops alone produce a comparatively high ultrafiltration. If (A.19) holds, a high ultrafiltration can, of course, be generated by means of a sufficient $P_{tm}$ even then, $q_u(x)$ is a nearly linear function.

When $q_u(x)$ is nearly linear, one may write

$$q_u = m - nx,$$

(A.20)

in which, from (A.18), (A.17), (A.1) and (A.2),

$$m = \frac{U P_{tm}}{L},$$

(A.21)

$$n = \frac{U}{L^2} \left( \frac{Q_b^O + Q_d^O}{\sqrt{\frac{a}{b}} + \frac{u}{b}} \right),$$

(A.22)
wherein \( \Delta p_d \) and \( \Delta p_d \) are the actual pressure drops.

Typical values of \( \sqrt[3]{\ell(R_0 + R_d)} \) for common dialyzers are in the order of 0.1. Common constructions with high-flux membranes may have values up to an order of 0.25. Hemofilters and plasmapheresis filters have still higher values.

APPENDIX 2

The membrane permeability with and without ultrafiltration

As shown in [1], the total solute flux \( J_S \) through a flat membrane under ultrafiltration may be written as (using notations of the preceding analysis)

\[
J_S = k_0(C_1 - C_2) + \frac{q_u L}{A} C^c, \tag{A.23}
\]

wherein

\[
C^c = C_1 - (C_1 - C_2) \left( 1 - \frac{1}{\Theta} \right), \tag{A.24}
\]

in which

\[
\Theta = \frac{q_u L S}{A \Theta_0} - 1 \tag{A.25}
\]

\( C_1 \) is here the concentration at the inside of the membrane and \( C_2 \) at the outside, in relation to the direction of \( J_S \) — not including boundary layers.

Rearranging (A.23), one finds

\[
J_S = k(C_1 - C_2) + \frac{q_u L}{A} C_1, \tag{A.26}
\]

wherein, for a flat membrane,

\[
k = \frac{q_u L S}{A \Theta_0 (e^{\Theta_0} - 1)} \tag{A.27}
\]

is the actual diffusion permeability factor at the inside of the membrane (cf. Appendix 3).

For small values of \( \Theta_0 \), one finds

\[
k \approx \frac{2k_0^2 A}{2k_0 A + q_u L S}, \tag{A.28}
\]

which approaches \( k_0 \) as \( q_u \to 0 \).

The expression (A.26) appears physically more appropriate than (A.23), since \( k(C_1 - C_2) \) is the portion of the solute extracted from the blood through diffusion, and \( \frac{q_u L S}{A} \) the portion extracted through convection — as seen from the blood side. Inside the membrane, the partition between diffusive and convective portions is gradually shifted.

From [2] (where the study is done for \( S = 1 \)), one finds for a hollow fiber, introducing the sieving coefficient \( S \)

\[
k = \frac{q_u L S}{A_1} \left( 1 - \frac{1}{A_1 k_0} \right) \tag{A.29}
\]

\[
\left( \frac{r_2}{r_1} \right)^2 A_1 D_w = 1
\]

wherein \( A_1 \) is the inner surface area of the fiber bundle and \( D_w \) the apparent diffusion coefficient in the fiber wall (as determined from external surface concentrations). \( r_1 \) is the inner and \( r_2 \) the outer radius of the wet fiber.

At \( q_u = 0 \), the permeability factor is, according to [2],

\[
k_0 = \frac{D_w}{r_1 \ln \frac{r_2}{r_1}}, \tag{A.30}
\]

Inserting this in (A.29), one finds

\[
k = \frac{q_u L S}{A_1 (e^{\Theta_0} - 1)} \tag{A.31}
\]

As \( r_1 \to \infty \), this expression approaches (A.27), since then \( A_1 \to \infty \) (vide infra); at the same time, \( k_0 \) approaches

\[
\lim_{r_1 \to \infty} k_0 = \frac{D_w}{\delta}, \tag{A.32}
\]

wherein \( \delta = r_2 - r_1 \) is the membrane thickness. (A.32) is the relation for a flat membrane.

As is shown in [3], the inner surface area \( A_1 \) of the fiber bundle relates to its true surface area \( A \) as

\[
A = 2\pi r_1 L, \tag{A.33}
\]

\[
A_1 = 2\pi r_1 L, \tag{A.34}
\]

\[
A = 2\pi r_1 L, \tag{A.35}
\]

\[
A_1 = \frac{A_1}{A_1 (e^{\Theta_0} - 1)} \tag{A.36}
\]

wherein \( N \) is the number of fibers in the bundle.

APPENDIX 3

Combination of membranes

In the case of zero ultrafiltration, the combined effect of stacked membranes is easily found. The total diffusion resistance (the inverted total permeability factor) is simply the sum of the individual diffusion resistances. There-
fore, the combined effect of one membrane and the boundary layers (treated as membranes) is found by means of a simple process of superposition.

In the presence of ultrafiltration, things become a bit more complicated. Nevertheless, a kind of superposition process can be devised.

In Appendix 2, the solute flux $J_b$ was written as

$$J_b = k(C_1 - C_2) + \frac{q_u L}{A} SC_1,$$  \hspace{1cm} (A.26)

wherein the diffusive portion is expressed as seen from the blood side. Analogously, one may rearrange to

$$J_b = k_d(C_1 - C_2) + \frac{q_u L}{A} SC_2,$$  \hspace{1cm} (A.37)

wherein the diffusive portion is expressed as seen from the dialysate side. One here finds

$$\frac{q_u L S}{A k_o},$$  \hspace{1cm} (A.38)

with $k$ according to (A.27), and notes that

$$k_d = k - \frac{q_u L S}{A},$$  \hspace{1cm} (A.39)

i.e., the difference amounts to the convective permeability factor, as seen from either side.

One may also rearrange as

$$J_b = C_1 k_d - C_2 k,$$  \hspace{1cm} (A.40)

which expresses a kind of superposition of the diffusive contributions from the respective other side. The contribution from the blood side on the dialysate side is then expressed through $C_1 k_d$ (as if $C_2$ were zero), and the contribution from the dialysate side on the blood side is expressed through $-C_2 k$ (as if $C_1$ were zero). This can be developed into a superposition principle for stacked membranes. The influence of convection is in (A.40) implicitly included according to (A.39).

If two flat membranes are stacked together, one can calculate the concentration $C_2$ at their common contact surface (interface) by means of mass balance, equating the output solute flux from the one membrane with the input solute flux to the next, at that surface. This way, one finds the overall diffusive permeabilities

$$k = \frac{k_1 k_2}{k_1 + k_2 d},$$  \hspace{1cm} (A.41)

and

$$k_d = \frac{k_1 d k_2 d}{k_1 + k_2 d},$$  \hspace{1cm} (A.42)

where, again, sieving coefficients are implicitly included. Indices 1 and 2 denote the individual membranes as numbered from the blood side.

If $v$ membranes are stacked, numbered 1, ..., $v$ from the blood side, the following formulae can be shown to be valid by means of induction (treating the stack from 1 to $v - 1$ as one membrane and membrane $v$ as the other):

$$k = \frac{1}{\Sigma j=1^v \Pi k_p},$$  \hspace{1cm} (A.43)

$$k_d = \frac{1}{\Sigma j=1^v \Pi k_d},$$  \hspace{1cm} (A.44)

wherein

$$\Sigma = \sum_{j=1}^{v-1} k_j + \sum_{i=1}^{v-1} \left( \sum_{j=i}^{v-1} k_j \prod_{j=i+1}^{v} k_d \right),$$  \hspace{1cm} (A.45)

with the convention

$$\prod_{j=\mu+1}^{v} z_j = 1$$  \hspace{1cm} (A.46)

for all $\mu$ (especially for $\mu = 0$ and $\mu = v$) and any $z_j$.

From this, one finds, with (A.39),

$$k_d = k - \frac{q_u L S}{A}$$  \hspace{1cm} (A.47)

for the total sieving coefficient $S$.

With (A.38) for the individual membranes, the difference between the products in (A.47) can also be written as

$$\prod_{j=1}^{v} k_{d,j} - \prod_{j=1}^{v} k_{j} =$$

$$= \frac{q_u L}{A} \sum_{i=1}^{v} \prod_{j=1}^{i-1} k_j \prod_{j=i+1}^{v} k_{d,j} S_i,$$  \hspace{1cm} (A.48)

so that one finds

$$S = \frac{1}{\Sigma} \sum_{i=1}^{v} \left( \prod_{j=1}^{i-1} k_j \prod_{j=i+1}^{v} k_{d,j} S_i \right).$$  \hspace{1cm} (A.49)

Applying (A.38) in an analogous manner, one finds

$$k_d = \frac{q_u L}{A} \sum_{j=1}^{v} \frac{S_j}{k_{o,j}} = e \frac{q_u L S'}{A k_o},$$  \hspace{1cm} (A.50)

so that

$$S' = \sum_{j=1}^{v} \frac{S_j}{k_{o,j}}.$$  \hspace{1cm} (A.51)

Hence, for $q_u \neq 0$, $S' \neq S$ in the general case. They become equal in the limit as $q_u \to 0$. Therefore, we can approximate $S \approx S'$ for small $q_u$, from which we conclude that (A.27) and (A.28) can be approximately applied for the total parameters of the stack of membranes, i.e., for the equivalent parameters, taken as one membrane.
Hence for small values of \( Q_U \) we can use (A.51) and (A.27) or (A.28) with
\[
\frac{1}{k_0} = \sum_{j=1}^{\infty} \frac{1}{k_{0j}},
\]
which is the relation for the total permeability factor at zero ultrafiltration.

In the case of tubular membranes stacked radially, things are a bit more complicated still, since the areas of the contact surfaces are no more the same. The solute flux \( J_{sr1} \), referred to the inner radius \( r_1 \) of the tubular membrane, is, according to Appendix 2,
\[
J_{sr1} = k(C_1 - C_2) + \frac{\varphi_{UL}}{A} \frac{S C_1}{k_d C_1 - k_d C_2},
\]
with \( k \) according to (A.31). From this one finds
\[
\frac{\varphi_{UL}}{A} = \frac{k_d}{k_0},
\]
and
\[
k_d = k \frac{q_{UL} S}{A_1},
\]
wherein \( k \) and \( k_d \) are both referred to \( r_1 \).

The solute flux at any other radius in the membrane is
\[
J_{sr} = \frac{r}{r_1} J_{sr1},
\]
as follows from the law of continuity. The diffusive permeabilities change accordingly and become, as referred to \( r \),
\[
k_r = \frac{r}{r_1} k_c,
\]
\[
k_d r = \frac{r}{r_1} k_d r,
\]
If one stacks \( n \) membranes radially, having the inner radii \( r_1, r_2, \ldots, r_n \), one finds in the same manner as above:
\[
\Sigma^{k_1} = \frac{1}{\Sigma} \sum_{j=1}^{\infty} k_{r_1} j_{r_1},
\]
\[
k_{dr} = \frac{1}{\Sigma} \sum_{j=1}^{\infty} k_{dr_1} j_{dr_1},
\]
and
\[
\Sigma = \sum_{i=1}^{\infty} \left( \prod_{j=1}^{i-1} k_{r_1} j_{r_1} \right) \frac{1}{i!(i+1)!} k_{dr_1}
\]
with the convention (A.46). Furthermore:
\[
\frac{1}{k_0} = \sum_{j=1}^{\infty} \frac{1}{k_{0j}},
\]
whereas (A.51) remains unchanged. Again, for small \( Q_U \), one may as an approximation use (A.51) and (A.27) or (A.28) with (A.63).

In (A.59) — (A.63) and in (A.51) for tubular membranes, \( k_1 \) and \( k_d \) are always referred to \( r_1 \).

Remark concerning modelling of mass transport in membranes and related experimental studies

It seems that equation (A.40) should be in a suitable general form for description of mass transport through membranes. Various studies have been published, such as in [1] and [6], giving different results for \( \varphi \) in (A.23). Equation (A.40) applies in both cases, but with different flow dependencies for \( k \) and \( k_d \), as functions of the ultrafiltration flow. It should be possible to separately measure \( k \) and \( k_d \) (or to separate them out from sets of measurements), using suitable experimental arrangements, as functions of \( Q_U \). In this way, theories on transport in membranes could be experimentally tested in a relatively strict manner.

Note

In physical membranes, the concentration usually suddenly jumps from the outside value in the immersing liquid just at the membrane surface to a different inside value in the membrane, just inside the surface. However, one may calculate with equivalent values of concentrations in the membrane, neglecting such jumps and equating outside and inside concentrations at the membrane surface. As a result, apparent values of diffusion constants and permeabilities apply, relating to concentration values outside the membrane, just at its surfaces. Concentrations, permeabilities and diffusion constants above which pertain to membranes are to be understood as such equivalent or apparent values. It is anyway the apparent permeability which is measured in the first place, when evaluating membranes — true permeabilities have to be estimated from estimates of inside concentrations, usually difficult to measure.

Notations

- \( a \) = a constant
- \( A \) = total membrane area
- \( A_1 \) = inner membrane area of hollow-fiber bundle
- \( A_b \) = total cross-section area of hollow-fiber bundle
- \( b \) = constant
- \( C \) = mean concentration in membrane
- \( C_b \) = blood concentration in solute
- \( C_b(s) \) = \( C_b \) at \( x = L \)
- \( C_b(s) = C_b \) at \( Q_U = 0 \)
- \( C_d \) = dialysate concentration of solute
- \( C_0 \) = \( C_b \) at \( x = O \)
- \( C_1 \) = concentration at inside of membrane under solute flux \( J_s \)
Dialyzer clearance

\[
\begin{align*}
C_2 & = \text{concentration on outside of membrane under solute flux } J_s \\
d & = \text{dialysate channel height} \\
D_p & = \text{diffusion constant in blood} \\
D_d & = \text{diffusion constant in dialysate} \\
D_w & = \text{apparent diffusion constant in hollow-fiber wall} \\
f & = \text{function defined by (11)} \\
g & = \text{function defined by (13)} \\
h & = \text{blood channel height} \\
j & = \text{multiplication and summation index} \\
J_s & = \text{solute flux through membrane} \\
J_{sr} & = \text{solute flux through hollow-fiber wall at radius } r \\
J_{r1} & = \text{solute flux through hollow-fiber wall at radius } r_1 \\
k & = \text{permeability factor as seen from blood side of membrane} \\
k' & = \text{combined permeability factor for membrane and dialysate boundary layer} \\
k_b & = \text{permeability factor for blood boundary layer} \\
k_d & = \text{permeability factor as seen from dialysate side of membrane} \\
k_D & = \text{permeability factor for dialysate boundary layer} \\
k_{dr} & = k_d \text{ referred to radius } r \text{ in hollow fiber} \\
k_{di,j} & = k_d \text{ of membrane } i \text{ or } j \\
k_{ij} & = k \text{ of membrane } i \text{ or } j \\
k_r & = k \text{ referred to radius } r \text{ in hollow fiber} \\
k_Q & = k \text{ at } Q_U = 0 \\
k_0 & = k_0 \text{ of membrane } j \\
k_0L & = k_0L \\
L & = \text{active length of dialyzer membrane} \\
m & = \text{a constant} \\
n & = \text{a constant} \\
N & = \text{number of fibers in hollow-fiber bundle} \\
p & = \text{variable in (57)} \\
P_{tm} & = \text{transmembrane pressure} \\
P_{tmn} & = \text{mean (over } x \text{) transmembrane pressure} \\
P_{tmn0} & = \text{first positive root of equation (57)} \\
Q & = \text{axial density of ultrafiltration flow} \\
Q_B & = \text{blood flow} \\
Q_{b0} & = \text{at } Q_U = 0 \\
Q_{bL} & = \text{at } Q = L \\
Q_{c} & = \text{dialyzer clearance} \\
Q_d & = \text{dialysate flow} \\
Q_{d0} & = \text{at } Q = 0 \\
Q_{dL} & = \text{at } Q = L \\
Q_{U} & = \text{ultrafiltration flow} \\
r & = \text{radius in hollow fiber (from center)} \\
r_f & = \text{inner radius of fiber wall} \\
r_i & = \text{outer radius of fiber wall} \\
r_B & = \text{blood-path flow resistance} \\
r_D & = \text{dialysate-path flow resistance} \\
r_s & = \text{sieving coefficient} \\
S_1 & = S \text{ of membrane } i \text{ or } j \\
S & = \text{a kind of sieving coefficient defined by (A.50)} \\
S^* & = \text{parameter defined by (53)} \\
S^t & = \text{factor in (A.7)} \\
S^u & = \text{ultrafiltration coefficient of membrane} \\
V & = \text{function defined by (51)} \\
W & = \text{parameter defined by (58)} \\
X & = \text{function defined by (52)} \\
Y & = \text{axial coordinate, i.e., coordinate in blood-flow direction along membrane} \\
Z & = \text{variable in (A.46)} \\
\alpha & = \text{parameter defined by (56)} \\
\beta & = \text{a constant} \\
\delta & = \text{membrane thickness} \\
\Delta & = \text{change or deviation in subsequent quantity} \\
\Delta P_B & = \text{pressure drop over } L \text{ in blood path} \\
\Delta P_d & = \text{pressure drop over } L \text{ in dialysate path} \\
\Delta P_{d0} & = \text{at } Q_U = 0 \\
\eta & = \text{integration variable} \\
\phi & = \text{parameter defined by (A.25)} \\
\lambda & = \text{coefficient in (59)} \\
\mu & = \text{parameter in (A.46)} \\
\nu & = \text{number of stacked membranes} \\
\Sigma & = \text{sum defined by (A.45) for flat membranes and by (A.61) for hollow-fiber membranes} \\
\psi & = \text{integration variable} \\
\phi & = \text{value defined by (33)} \\
\end{align*}
\]

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This address is no more valid!

REFERENCES