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A THEORETICAL STUDY OF CAPACITIVE PLETHYSMOGRAPHY

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Abstract—A theoretical analysis of capacitive plethysmography is given which is more accurate than that from a previous study due to a more exact mathematical approach. As a result, it is shown that capacitive plethysmography can, under certain circumstances, be quite inaccurate. The mathematical techniques developed may, for each practical case, be used to improve accuracy.

NOTATION

- d = distance between electrode surfaces at right angles to both
- d_0 = constant part of varying d
- n = relative measure of a part of the perimeter p ($n = 1$ for the whole perimeter) or of a part of the axial length L
- p = perimeter of the inner electrode surface in a cross section at right angles to the axis
- r_i = radius of a circularly cylindrical inner electrode surface
- r_0 = radius of a circularly cylindrical outer electrode surface
- s = curvilinear co-ordinate along the surface of the inner electrode in a plane through the axis
- x = co-ordinate along the axis of the electrode surfaces
- C = capacitance of the electrode arrangement
- L = length of the axis of the electrode surfaces
- R = radius of curvature of the inner electrode surface in a plane through the axis
- R_p = radius of curvature of inner electrode surface in a cross section at right angles to the axis
- R_y = resistance associated with Y
- S = curve length along the surface of the inner electrode in a plane through the axis
- V = tissue volume within the inner electrode
- V_i = interspace volume between the electrode surfaces
- Y = admittance between connections to the electrode arrangement
- Y_q = quadrature component of Y
- α = angle between the inner electrode surface and the axis
- δ = distance between the inner electrode surface and a thought infinitesimal layer between the electrodes—also used for the varying part $d-d_0$ of d
- ϵ = relative dielectric constant
- ϵ_0 = absolute dielectric constant
- ξ = δ/d = relative value of δ
- ρ = resistivity of the tissue
- $d\varphi$ = centre angle of circular segment associated with the radius of curvature R (by definition)
- $d\psi$ = centre angle of circular segment associated with the radius of curvature R_p (by definition)
- ω = angular frequency of current used to measure Y

A few additional notations are used in the Appendices and are defined there.

INTRODUCTION

PLETHYSMOGRAPHY, the recording of volume changes in tissue, is mostly used for the measurement of variations in the volume of a segment of a limb as an indirect measurement of total blood flow through the segment or the mean perfusion of its tissue. One way to measure such a volume variation is to immerse the segment in a fluid and measure the variation in the resulting displacement. This is, for several reasons, still the method of choice if the best possible accuracy is required (direct transmission of volume change to transducer, direct calibration, etc., SIGDELL, 1968). On the other hand, working with a liquid involves a considerable complication of the practical procedure and some other methods have therefore been developed. Of those, the most convenient to use are the mercury strain gauge plethysmograph and the capacitive plethysmograph. A theoretical study of the former has been published (SIGDELL, 1969) and shows it to have a limited accuracy. Here the latter method now will be analysed.

THE CAPACITIVE PLETHYSMOGRAPH

The principle of this plethysmograph, developed originally by FIGAR (1959a and b) and further by HYMAN *et al.* (1964) and WOOD *et al.* (1970) is to use the surface of the limb as one electrode of a capacitor, the other being a metal cuff surrounding it. Obviously, the capacitance of this arrangement varies with the volume of the limb segment. It is here the purpose to develop a mathematical theory and to briefly study how accurately a volume change can be related to a corresponding capacitance change. This has been studied before (WILLOUGHBY, 1965), but, in the present authors opinion, that study gives an incorrect picture of the accuracy of the capacitive plethysmograph. The possible errors estimated by WILLOUGHBY (1965) are related to the *total* capacitance instead of to the capacitance *change* recorded (cf. WOOD *et al.*, 1970, Appendix 2), which is the relevant quantity. Furthermore a different mathematical approach has been chosen, as it provides a more exact theory.

A GENERAL FORMULA FOR THE CAPACITANCE BETWEEN CYLINDRICAL SURFACES AT A CONSTANT SEPARATION

Suppose a cylindrical surface with the perimeter p and length L to be surrounded by another cylindrical surface of the same length, such that the separation between them is everywhere constant, measured at right angles to both. If the inner surface is everywhere, convex, this arrangement is always possible. If the inner surface has concave portions, this is still possible if its inward radius of curvature is nowhere less in magnitude than the separation between the surfaces (i.e., generally, if $R_p \geq -d$ if R_p denotes the radius of curvature and d the distance so that $R_p > 0$ outwards and $R_p < 0$ inwards). As is shown in Appendix 1, the perimeter of the outer surface is then $p + 2\pi d$. Now consider two surfaces at distances δ and $\delta + \mathrm{d}\delta$ from the inner surface. The assumption that those are approximate equipotential surfaces is, in view of the result, justified by the fact that the true field distribution is an extremal function to the field energy as a functional of the set of fields. Hence a minor distortion of the field causes a negligible change in capacitance (compare the change of a function for a small deviation from an inner point of extremity). The capacitance of the layer between those two surfaces is given by

$$\mathrm{d}\left(\frac{1}{C}\right) = \frac{\mathrm{d}\delta}{L\epsilon\epsilon_0(p + 2\pi\delta)}, \tag{1}$$

hence the total capacitance is obtained by integration over the distance d between the surfaces:

$$\frac{1}{C} = \int \mathrm{d}\left(\frac{1}{C}\right) = \frac{1}{L\epsilon\epsilon_0}\int_0^d \frac{\mathrm{d}\delta}{p + 2\pi\delta}, \tag{2}$$

or

$$C = \frac{2\pi\epsilon\epsilon_0 L}{\ln\left(1 + 2\pi\frac{d}{p}\right)}. \tag{3}$$

Here ϵ is the relative and ϵ_0 the absolute dielectric constant. This expression reduces to the well known formula for circular cylinders if the inner (and therefore the outer) surface is circular. In Appendix 2 this expression is again derived in a way based on the reasoning by WILLOUGHBY (1965), showing consistency with his work, so far.

The end effects of the capacitor arrangement have not been considered as they could—and should—be eliminated by a "guard-ring" arrangement (see below).

EXPRESSION FOR NON-CYLINDRICAL SURFACES AT VARYING SEPARATIONS

If axial field components are neglected, equation (3) will hold for each segment of length dx, where x is the co-ordinate along the axis of the surfaces, provided that in each plane through this axis the distances from the axis to the surfaces vary slowly with x, i.e. the surfaces form small angles with the axis at each point, and that the distance d between the surfaces varies slowly with x and is constant along the perimeter (semi-cylindrical arrangement). The neglect of field components parallel to the surfaces is again justified by the fact that the true field is an extremal function to field energy as a functional of the set of field distributions. The minor distortion of the field through neglecting the axial component in a case like this causes a negligible change in capacitance. Therefore we obtain from (3), applied to the segment dx, after integration:

$$C = 2\pi\epsilon\epsilon_0 \int_0^L \frac{\mathrm{d}x}{\ln\left[1 + 2\pi\dfrac{d(x)}{p(x)}\right]}, \qquad (4)$$

where L is the length of the semi-cylindrical arrangement.

A more general relationship can be derived as follows. As a first step, consider non-cylindrical electrodes with rotational symmetry (circular cross-sections)—in a second step the full generalization will be carried out. Consider a segment of the electrode arrangement as drawn in Fig. 1 in an axial cross-section. The

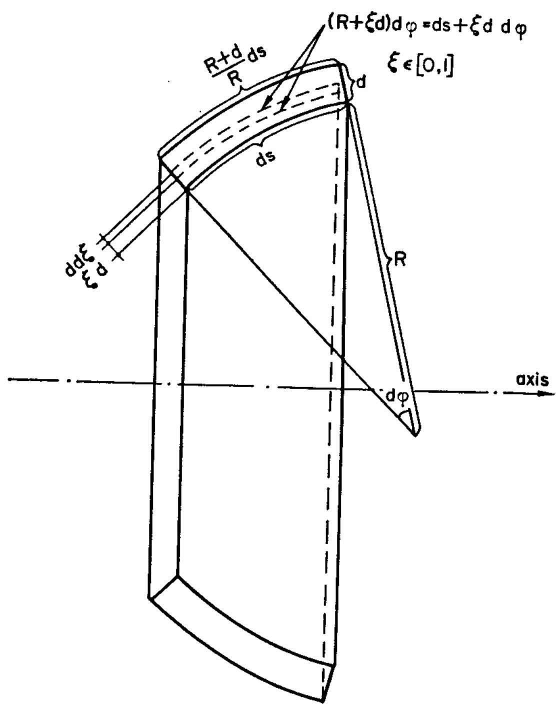

FIG. 1. An axial cross-section of a segment at rotational symmetry. R = radius of curvature.

curve length along the inner electrode (the limb surface) is ds and its radius of curvature in the plane through its axis is R. The capacitance of the thin layer of thickness $d\,\mathrm{d}\xi$ is (equipotential surface assumption justified as before, the distance d is assumed constant along p, i.e. in each cross-section cut at right angles to the axis)

$$\frac{1}{\mathrm{d}\left(\dfrac{1}{\mathrm{d}C}\right)} = \epsilon\epsilon_0 \frac{1}{d\,\mathrm{d}\xi}(\mathrm{d}s + \xi d\,\mathrm{d}\varphi)(p + 2\pi\xi d) \qquad (5)$$

with notations from Fig. 1. First invert and integrate with respect to ξ from 0 to 1:

$$\frac{1}{\mathrm{d}C} = \frac{1}{\epsilon\epsilon_0}\frac{1}{2\pi\,\mathrm{d}s - p\,\mathrm{d}\varphi}\ln\frac{(p + 2\pi d)\,\mathrm{d}s}{p(\mathrm{d}s + d\,\mathrm{d}\varphi)}. \qquad (6)$$

Invert again and integrate with respect to s from 0 to S, the total curve length along the inner electrode:

$$C = 2\pi\epsilon\epsilon_0 \int_0^S \frac{1 - \dfrac{p}{2\pi R}}{\ln\dfrac{R(p + 2\pi d)}{p(R + d)}}\,\mathrm{d}s, \qquad (7)$$

as $R = \mathrm{d}s/\mathrm{d}\varphi$. For $R \gg d$ we may write this as

$$C = 2\pi\epsilon\epsilon_0 \int_0^S \frac{1 - \dfrac{p}{2\pi R}}{\ln\left(1 + 2\pi\dfrac{d}{p}\right) - \dfrac{d}{R}}\,\mathrm{d}s. \qquad (8)$$

In general, p, R and d vary with s. This expression reduces further to (4) if $R \gg p$ and $x \gg s$. In fact it shows (4) to be valid also for a (nearly) conical arrangement—if x is substituted by s and L by S—or, more generally, for an arrangement with rotational symmetry and much less curvature in an axial direction than in a corresponding cross-section.

Now, as a final step towards a general expression, allow for an arbitrary cross-section. We should then write, instead of (5),

$$\mathrm{d}\left[\frac{1}{\mathrm{d}\left(\dfrac{1}{\mathrm{d}C}\right)}\right] = \epsilon\epsilon_0 \frac{1}{d\,\mathrm{d}\xi}(\mathrm{d}s + \xi d\,\mathrm{d}\varphi)(\mathrm{d}p + \xi d\,\mathrm{d}\psi), \qquad (9)$$

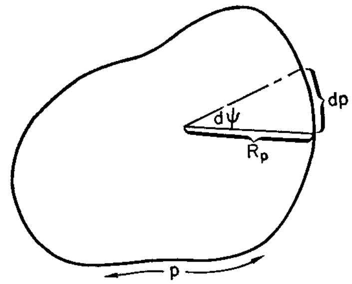

FIG. 2. A cross-section of the "inner electrode" at right angles to the axis, no rotational symmetry. R_p = radius of curvature.

where $\mathrm{d}\psi$ is a sector of the cross-section as shown in Fig. 2, having the p-contour's local radius of curvature R_p as sides. (9) must then be integrated over the p-curvature, which makes ψ vary from 0 to 2π. The expression corresponding to (6) becomes, as $R_p = \mathrm{d}p/\mathrm{d}\psi$,

$$\mathrm{d}C = \epsilon\epsilon_0 \oint \left[\mathrm{d}s \frac{1/R_p - 1/R}{\ln \dfrac{1 + d/R_p}{1 + d/R}} \right] \mathrm{d}p, \qquad (10)$$

where generally also $\mathrm{d}s$ varies along p. To integrate in the axial direction, we have to introduce the angle α between the surface and the axis and express $\mathrm{d}s$ as $\mathrm{d}x/\cos\alpha$, as there is no general S as an upper boundary of an integral like in (7). Hence

$$C = \epsilon\epsilon_0 \int_0^L \oint \frac{R - R_p}{RR_p \cos\alpha \ln \dfrac{1 + d/R_p}{1 + d/R}} \mathrm{d}p \, \mathrm{d}x, \qquad (11)$$

where the closed path integral is taken over the closed p-contour in a cross-section. In general, R, R_p, α and d vary with p and x.

In the case $R, R_p \gg d$ we can considerably simplify (11) by retaining only the first term in the series expansion of the logarithm:

$$C = \epsilon\epsilon_0 \int_0^L \oint \frac{\mathrm{d}p \, \mathrm{d}x}{d \cos\alpha}. \qquad (12)$$

For many applications, the condition for this simplified expression should not be difficult to realize in practice.

THE INTERSPACE VOLUME BETWEEN THE TWO SURFACES AND ITS RELATION TO CAPACITANCE

Referring to Figs. 1 and 2, we find for the interspace volume V_i:

$$\mathrm{d}^3 V_i = d \, \mathrm{d}\xi (\mathrm{d}s + \xi d \, \mathrm{d}\varphi)(\mathrm{d}p + \xi d \, \mathrm{d}\psi), \qquad (13)$$

or, as $R = \mathrm{d}s/\mathrm{d}\varphi$ and $R_p = \mathrm{d}p/\mathrm{d}\psi$:

$$\mathrm{d}^2 V_i = d \left[1 + \frac{1}{2}\left(\frac{d}{R} + \frac{d}{R_p} \right) + \frac{1}{3} \frac{d^2}{RR_p} \right] \mathrm{d}p \, \mathrm{d}s, \qquad (14)$$

from integrating (13) between $\xi = 0$ and $\xi = 1$. Again we set $\mathrm{d}x = \mathrm{d}s \cos\alpha$:

$$V_i = \int_0^L \oint \frac{d}{\cos\alpha} \left[1 + \frac{d}{2}\left(\frac{1}{R} + \frac{1}{R_p} \right) + \frac{1}{3} \frac{d^2}{RR_p} \right] \mathrm{d}p \, \mathrm{d}x. \qquad (15)$$

If R, $R_p \gg d$ this reduces to

$$V_i = \int_0^L \oint \frac{d}{\cos\alpha} \mathrm{d}p \, \mathrm{d}x, \qquad (16)$$

or, with (12), for a constant d:

$$C_i = \frac{\epsilon\epsilon_0}{d^2} V_i, \qquad (17)$$

i.e. an ideal, linear relationship between capacitance and volume.

For a *small* change Δd in d, corresponding to a change $\Delta V = -\Delta V_i$ in the volume V of the limb segment, we have in the case R, $R_p \gg d$ if $\cos\alpha$ remains unchanged (this is a plausible assumption, especially if α is small):

$$\left\{ \begin{aligned} \Delta V_i &= \int_0^L \oint \frac{\Delta d}{\cos\alpha} \mathrm{d}p \, \mathrm{d}x, & (18) \\ \Delta C &= -\epsilon\epsilon_0 \int_0^L \oint \frac{1}{d^2} \frac{\Delta d}{\cos\alpha} \mathrm{d}p \, \mathrm{d}x, & (19) \end{aligned} \right.$$

or, for a constant d

$$\Delta C = -\frac{\epsilon\epsilon_0}{d^2} \Delta V_i = \frac{\epsilon\epsilon_0}{d^2} \Delta V, \qquad (20)$$

independently of the distribution of Δd along the surface.

More exactly, we have

$$\Delta V_i = \int_0^L \oint \frac{\Delta d}{\cos \alpha} \left[1 + d\left(\frac{1}{R} + \frac{1}{R_p}\right) + \frac{d^2}{RR_p}\right] \mathrm{d}p\, \mathrm{d}x, \quad (21)$$

$$\Delta C = -\epsilon\epsilon_0 \int_0^L \oint \left(\frac{R - R_p}{RR_p \ln \dfrac{1 + d/R_p}{1 + d/R}}\right)^2 \frac{\Delta d . \mathrm{d}p\, \mathrm{d}x}{\left(1 + \dfrac{d}{R}\right)\left(1 + \dfrac{d}{R_p}\right) \cos \alpha}. \quad (22)$$

If d is still small, compared to R and R_p we may estimate the error in (20) by expanding (22) in series and only keeping first order terms in d:

$$\begin{cases} \Delta V_i = \displaystyle\int_0^L \oint \frac{\Delta d}{\cos \alpha} \left[1 + d\left(\frac{1}{R} + \frac{1}{R_p}\right)\right] \mathrm{d}p\, \mathrm{d}x, & (23) \\ \Delta C = -\epsilon\epsilon_0 \displaystyle\int_0^L \oint \frac{\Delta d}{\cos \alpha} \left[1 - 5d \left(\frac{1}{R} + \frac{1}{R_p}\right)\right] \mathrm{d}p\, \mathrm{d}x, & (24) \end{cases}$$

or

$$\Delta C = -\frac{\epsilon\epsilon_0}{d^2} \Delta V_i \left[1 - \frac{6}{\Delta V_i} \int_0^L \oint \frac{\Delta d}{\cos \alpha} \left(\frac{d}{R} + \frac{d}{R_p}\right) \mathrm{d}p\, \mathrm{d}x\right]. \quad (25)$$

We see that if R and R_p are constant, a linear relationship between ΔV_i and ΔC still holds, as is also the case if Δd is constant. If they are not constant, the relationship is no longer exactly linear and furthermore depends, to some extent, on the distribution of Δd over the surface. The more Δd concentrates on parts of the surface where R and R_p take minimum values, the worse are these effects. More general estimates are obtained by taking additional terms of the series expansion.

A general estimate of non-linearities and errors cannot be made as variations in R and R_p have to be specified. A study of the simple case of a circular cylinder is not of interest as there are no errors in that case (as can also be verified by applying known formulae directly). Note that the case where R and R_p are constant is linear, even if the true relationship deviates from (20), and is independent of the distribution of Δd. Therefore a possible deviation from (20) in this case is automatically compensated for by means of a suitable method of calibration (the most accurate calibration should be to inject blood or some compatible solution, or drain blood, to a known volume in the limb segment with the circulation arrested). If this is not done and the measurement interpreted according to (20), (25) shows the error to produce an underestimation of ΔV by a factor

$$1 - 6d \frac{1/R + 1/R_p}{1 + d(1/R + 1/R_p)}. \quad (26)$$

THE RELATIONSHIP FOR A NON-INFINITESIMAL CHANGE

We have hitherto studied the performance of the system for an approximately infinitesimal change Δd in d. If a larger change is considered, (18) still holds for the case R, $R_p \gg d$, but (12) gives for a constant d and a negligible change in $\cos \alpha$:

$$\Delta C = -\epsilon\epsilon_0 \int_0^L \oint \frac{\Delta d}{d(d + \Delta d) \cos \alpha} \mathrm{d}p\, \mathrm{d}x$$
$$= -\frac{\epsilon\epsilon_0}{d^2} \int_0^L \oint \frac{\Delta d}{\cos \alpha} \left[1 - \frac{\Delta d}{d} + \left(\frac{\Delta d}{d}\right)^2 - \cdots\right] \mathrm{d}p\, \mathrm{d}x$$
$$\approx -\frac{\epsilon\epsilon_0}{d^2} \Delta V_i \left[1 - \frac{\overline{\Delta d}}{d} + \left(\frac{\overline{\Delta d}}{d}\right)^2 - \cdots\right], \quad (27)$$

where $\overline{\Delta d}$ is an appropriately defined mean value of Δd.

In this case we find, due to the non-linearity arising with a larger Δd, a certain effect from the distribution of Δd. Especially, if we assume Δd to occur only along $1/n$ of the perimeter, as an illustrative example, we find (assume Δd constant there and independent of x):

$$\Delta C = -\frac{\epsilon\epsilon_0}{d^2}\,\Delta V_i\left[1 - n\frac{\Delta V_i}{V_i} + n^2\left(\frac{\Delta V_i}{V_i}\right)^2 - \cdots\right]. \tag{28}$$

A physiologically plausible order of magnitude of $\Delta V_i/V_i$ is 10^{-2} for perfusion measurement (from the initial derivative of the volume change) and at most 10^{-1} for, e.g., the study of the capillary filtration coefficient (SIGDELL, 1968). We see that, under certain circumstances, quite a large error may arise if ideal relationships are assumed. For example, for $n = 5$ and $\Delta V_i/V_i = 0{\cdot}04$ we find an error of about 20 per cent (for this case WILLOUGHBY, 1965, determines an error of less than 0·8 per cent in the *total* capacitance).

The study of errors can, of course, be extended to more general cases by using (15) and (22), similarly to the deduction of (25). This approach is rather elaborate in general terms and there is no reason to expect less error in such cases. Therefore we may feel satisfied with the indication of the accuracy of the capacitive plethysmograph obtained above.

DISCUSSION OF THE INFLUENCE OF ERRORS IN THE SPACING d

We again assume the relationships (18) and (19) to be valid with sufficient accuracy, i.e. that d is sufficiently smaller than R and R_p and that $\cos\alpha$ changes negligibly when the limb segment expands. We further assume a small error in the spacing d:

$$d = d_0 + \delta, \tag{29}$$

where d_0 is constant. We can then write, from (19),

$$\Delta C = -\epsilon\epsilon_0\int_0^L\oint\frac{\Delta d}{d_0{}^2\cos\alpha}\left(1-\frac{\delta}{d_0+\delta}\right)^2 \mathrm{d}p\,\mathrm{d}x$$

$$= -\frac{\epsilon\epsilon_0}{d_0{}^2}\,\Delta V_i + \frac{2\epsilon\epsilon_0}{d_0{}^2}\int_0^L\oint\frac{\delta\,\Delta d}{(d_0+\delta)\cos\alpha}\left(1-\frac{1}{2}\,\frac{\delta}{d_0+\delta}\right)\mathrm{d}p\,\mathrm{d}x. \tag{30}$$

If $\delta \ll d_0$, this simplifies to

$$\Delta C = -\frac{\epsilon\epsilon_0}{d_0{}^2}\,V_i + \frac{2\epsilon\epsilon_0}{d_0{}^3}\int_0^L\oint\frac{\delta\,\Delta d}{\cos\alpha}\,\mathrm{d}p\,\mathrm{d}x. \tag{31}$$

As an example, this is applied to a circular cylinder where δ is constant along $1/n$ of the perimeter and zero otherwise, independently of x. In the somewhat simpler case of (31) we then obtain, with (18):

$$\Delta C = -\frac{\epsilon\epsilon_0}{d_0{}^2}\,\Delta V_i\left(1-\frac{2\delta}{nd_0}\right). \tag{32}$$

This shows an error of about 8 per cent for $\delta/d_0 = 0{\cdot}2$ and $n = 5$* (WILLOUGHBY, 1965, finds for the same δ/d_0 an error, in *total* capacitance C, generally less than 2 per cent).

The same relationship (32) is easily derived, using the same assumptions of relative magnitudes, directly from the known formula for a cylindrical capacitor. In both cases angular field components have been neglected with the same justification as before, although this special case with a discontinuous outer electrode makes the situation somewhat more critical. However, this case is only a simplified model of the more natural case of a continuous δ and this way of studying it is still informative about the magnitude of the errors caused by δ.

The same relationship applies again if δ is constant along the perimeter and along $1/n$ of

* Equation (32) is actually a little too approximative for $\delta/d = 0{\cdot}2$; (30) gives an error of 6 per cent for this case, as does a calculation from the known formula for a cylindrical capacitor, assuming a radius $\gg d$ but not $\delta \ll d_0$.

the total length of the cylindrical arrangement, but zero elsewhere.

In general, it can be stated that if $\delta \ll d_0$ the error is small as long as (18) and (19) apply to a sufficient accuracy, as then δ and d add their contributions to capacitance in a linear manner.

ALLOWABLE SPACING

Firstly, the spacing must be sufficiently large to allow for the expected expansion without too much error in the determination of the volume change. But there are also electrical criteria for an allowable spacing.

The spacing d determines the basic capacitance of the arrangement. Depending upon the frequency used when evaluating the capacitance, there is a lower limit for acceptable capacitance values, i.e. a higher limit for acceptable d-values.

The admittance measured between the electrodes has a rather complicated expression as there is a distribution of resistance along the inner "electrode" (the skin). This can be estimated approximately using one resistance and the basic capacitance in series:

$$Y \approx \frac{1}{R_y}\frac{\omega^2 R_y^2 C^2}{1+\omega^2 R_y^2 C^2} + j\omega C\frac{1}{1+\omega^2 R_y^2 C^2}. \tag{33}$$

Preferably, the measurement should be performed in such a way that only the quadratic component of Y is read (e.g., by phase-sensitive detection):

$$Y_q \approx \frac{j\omega C}{1+\omega^2 R_y^2 C^2}. \tag{34}$$

Hence we should have

$$(\omega R_y C)^2 \ll 1 \tag{35}$$

to avoid resistive errors.

More generally we should have a small effect from the change in the resistance (under certain circumstances the resistance may change less than the capacitance), i.e. we should have

$$\Delta Y_q \approx j\omega \frac{\Delta C(1-\omega^2 R_y^2 C^2) - 2R_y\omega^2 C^3 \Delta R_y}{(1+\omega^2 R_y^2 C^2)^2}$$

$$\approx j\omega \frac{\Delta C(1-\omega^2 R_y^2 C^2)}{(1+\omega^2 R_y^2 C^2)^2}, \tag{36}$$

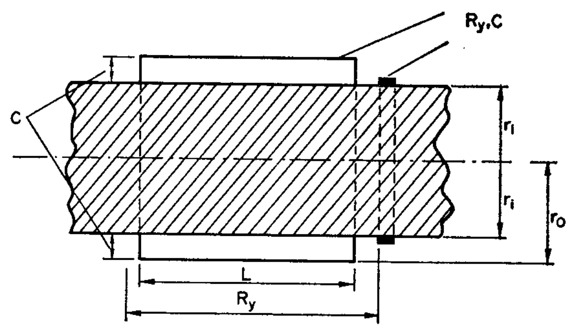

FIG. 3. Estimations of resistance and basic capacitance in a circularly cylindrical model.

where the denominator does not necessarily have to reduce to unity—if Y_q is proportional to ΔC but not influenced by ΔR_y a suitable method of calibration automatically compensates for the fact that the denominator is not unity. Equation (36) requires

$$|(1-\omega^2 R_y^2 C^2)\,\Delta C| \gg 2R_y\omega^2 C^3 |\Delta R_y|. \tag{37}$$

We remain "on the safe side" if we estimate the resistance as sketched in Fig. 3. For circular cylinders this gives (radii as defined in Fig. 3)

$$R_y \approx \rho\frac{L}{\pi r_i^2} = \frac{\rho L^2}{V}, \tag{38}$$

$$C = \frac{2\pi L\epsilon\epsilon_0}{\ln\frac{r_0}{r_i}} = \frac{4\pi L\epsilon\epsilon_0}{\ln\frac{\pi r_0^2 L}{V}}, \tag{39}$$

where V is the volume of the limb segment. In the case where (35) is valid (in order to simplify the otherwise somewhat complex expression) this gives with (37):

$$r_i^4 \ln\frac{r_0}{r_i} \gg 16\,\epsilon\epsilon_0(\omega\rho L^2)^2. \tag{40}$$

An estimate of an upper limit for $d = r_0 - r_i$ is hence given by (40) in this case. It is interesting to note that (35) leads to an expression just like (40), with (38) and (39), except for the factor on the right side, which is 4 instead of 16. Thus (40) is a somewhat more severe condition than (35), i.e. both are fulfilled if (40) is fulfilled.

Normally there is air as the dielectric, giving $\epsilon \approx 1$. Measurements show (GEDDES, 1967) that the mean resistivity ρ of the arm is about 330 Ω cm along the muscle fibres and about 470 Ω cm across them.

TISSUE SURFACE CONDITION

The penetration of the interelectrode electric field into the tissue depends on the humidity of the skin (surrounding humidity, perspiration) and should also, to some extent, depend on the perfusion of the skin (influenced by temperature and by physiological and even psychological factors). Therefore it is important that those conditions do not change during the measurement. Otherwise a change in, e.g., humidity (caused, for example, by perspiration) may cause a capacitance change which is interpreted as a volume change. The safest precaution should be to establish nearly 100 per cent humidity (small penetration depth) in the skin by a proper treatment. Perhaps electrode paste (to raise conductivity) and a thin rubber sleeve (to prevent loss through evaporation) might be used.

Evaporation from the skin may significantly alter the dielectric constant of the air between the electrodes. The formation of drops of perspiration may also be measured as an irrelevant volume expansion. A thin non-compressing rubber sleeve could prevent these influences as well.

END EFFECTS

These should be eliminated, together with the capacitance from the outer electrode to the exterior, by a combination of a "guard ring" and "hot shield" arrangement, as sketched in Fig. 4.

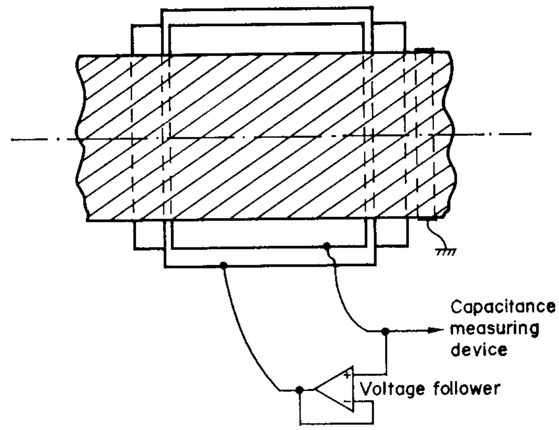

FIG. 4. The use of "guard rings" and a "hot shield".

CONCLUSION

It has been shown that the method of capacitive plethysmography, under certain circumstances, may give rise to considerable errors. On the other hand the method is handy and simple to use. The mathematical basis given here may possibly serve as a means for improving the result by actually calculating the errors and arranging to compensate for them. This is, however, difficult in general terms and requires a sufficiently accurate knowledge of the geometry for each special case as well as of the distribution of the expansion over the surface of the limb segment. Another possible use of this basis may be to calculate an optimum spacing, optimized in view of the increase in error with increasing d for geometrical reasons as opposed to the decrease in error with increasing d, related to the magnitude of the expected actual expansion. Such extensions of the theoretical analysis have been avoided here in order to keep the paper within reasonable limits. The purpose has been to develop the basic methods for such extensions and to briefly discuss the effects of errors.

The aim of this study has been to analyse in general terms the properties of a capacitive arrangement of two conductive surfaces, closed in a cross-section, and close to each other. The forms of these surfaces are not prescribed, so that the description is usable with any practical arrangement. The study is not related to any special capacitive plethysmograph, and is of a purely theoretical nature. It is aimed at including a basic approach to the analysis of this technique. (The practical problem of mounting the electrodes has not been discussed as it is not within the scope of this paper.) One approximation has been made, because it was necessary for practical reasons, in that the field components parallel to the surfaces are neglected. This is justified by reference to the theories of functionals (functions of functions) and the calculus of variations.

ADDENDUM

Note on two special cases

For most practical applications the radii of curvature fulfil

$$R \gg R_p.$$

If in such a case d is chosen so that

$$d = kR_p \ll R,$$

where k is a constant, we can write, from (21) and (22):

$$\Delta C = -\frac{\epsilon\epsilon_0}{(1+k)\ln^2(1+k)}\int_0^L \oint \frac{\Delta d\,\mathrm{d}p\,\mathrm{d}x}{R_p{}^2\cos\alpha}$$

$$= -\frac{\epsilon\epsilon_0 k^2}{(1+k)\ln^2(1+k)}\int_0^L \oint \frac{\Delta d\,\mathrm{d}p\,\mathrm{d}x}{d^2\cos\alpha}$$

and

$$\Delta V_t = (1+k)\int_0^L \oint \frac{\Delta d\,\mathrm{d}p\,\mathrm{d}x}{\cos\alpha}.$$

Hence formulae essentially identical to (18) and (19) apply in this case—the difference is only in the constants before the integrals. This means that the study of the simplified relations (18) and (19) in the paper can be generalized also to the case when $R \gg R_p$, d, but not necessarily $d \ll R_p$, if d is made to vary proportionally to R_p. This is an interesting and potentially important way to improve the accuracy of the capacitive plethysmograph considerably in many practical cases.

A further, but somewhat complicated, possibility to improve the accuracy is to make d vary with R_p as implicitly given by

$$R_p\left(1+\frac{d}{R_p}\right)\ln\left(1+\frac{d}{R_p}\right) = k,$$

where k is again a constant. In this case we find from (21) and (22), if $R \gg R_p$, d:

$$\Delta C = -\frac{\epsilon\epsilon_0}{k^2}\int_0^L \oint \frac{\Delta d}{\cos\alpha}\left(1+\frac{d}{R_p}\right)\mathrm{d}p\,\mathrm{d}x$$

and

$$\Delta V_t = \int_0^L \oint \frac{\Delta d}{\cos\alpha}\left(1+\frac{d}{R_p}\right)\mathrm{d}p\,\mathrm{d}x,$$

that is

$$\Delta C = -\frac{\epsilon\epsilon_0}{k^2}\Delta V_t,$$

an ideal linear relation for approximately infinitesimal increments.

For non-infinitesimal increments the first of those two improvements makes (27) hold for that case as well, except for another constant before the integral, which in many cases should be an improvement as the more complicated general relation corresponding to (27), which is to be derived from (11) and otherwise might apply instead, is not expected to offer less error for other than very special cases. Actually, the latter of the two cases above can be such a very special case, as we here find, corresponding to (27):

$$\Delta C = -\frac{\epsilon\epsilon_0}{k^2}\int_0^L \oint \frac{\Delta d}{\cos\alpha}\left(1+\frac{d}{R_p}\right)$$

$$\left\{1 - \frac{\Delta d}{k}\left[1 - \frac{1}{2}\ln\left(1+\frac{d}{R_p}\right)\right] + \cdots\right\}\mathrm{d}p\,\mathrm{d}x$$

and

$$\Delta V_t = \int_0^L \oint \frac{\Delta d}{\cos\alpha}\left(1+\frac{d}{R_p}\right)\left[1 + \frac{\Delta d}{2R_p\left(1+\frac{d}{R_p}\right)}\right]\mathrm{d}p\,\mathrm{d}x,$$

from which we see that it is possible to get a better accuracy than in the case of (27), under suitable circumstances.

Making d vary as a more complicated function of both R and R_p can further extend the validity of the simplified relations and reduce errors in still more general cases.

A comment

The paper by Wood and Hyman (1970) (read after the completion of this study) is an important contribution to capacitive plethysmography. The use of a flexible but non-elastic electrode, spaced from the limb by an elastic, insulating foam material, reduces errors from non-uniform expansion. In fact, their arrangement is equivalent to an arrangement with a rigid electrode but subject to a uniform expansion with the same volume change as with the flexible electrode (even if the expansion is not uniform in the latter case). The flexible electrode deforms slightly so as to equalize the expansion. The influence of the deformation should be of a second-order kind so that the above theory could be applied by simply assuming a uniform expansion and neglecting the effect of the change of the form of the electrode.

The plethysmograph of Wood and Hyman (1970) is, though, still subject to the influences of the radii of curvature. Such influences could be reduced by using a varying spacing, as outlined above. It should be possible to do this in a fairly general way by having a foam sleeve with varying thickness, designed for an "average arm" or an "average leg". The elasticity of the foam material must, of course, be such that it gives rise to a negligible pressure on the limb (especially as this varies with the expansion).

The theory presented could also be modified so that it would apply for an elastic electrode.

APPENDIX 1

Consider a sector of the cross-section, formed by a short piece of the electrode surfaces and their radii of curvature—which meet in the same point under the same angle $d\varphi$ as d is assumed constant—as shown in Fig. 5. With the notations from the figure, we have:

$$ds_2 = (R_p + d)\,d\varphi,$$
$$ds_1 = R_p\,d\varphi,$$

hence

$$ds_2 - ds_1 = d\,d\varphi$$

and, integrating over φ,

$$s_2 - s_1 = 2\pi d$$

as obviously

$$\oint d\varphi = 2\pi.$$

This is easily checked by studying an arbitrary polygon as the cross-section of the inner surface. The outer contour then has straight parts of the same total length as the inner contour but also circle sectors with a total angle of 2π.

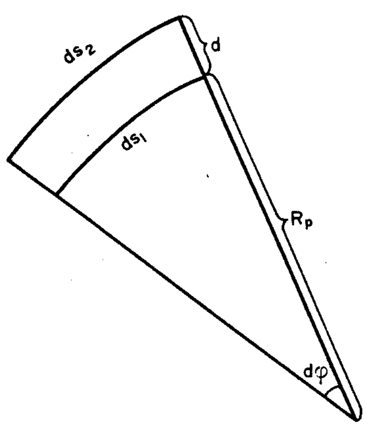

FIG. 5. Illustration to the derivation of the relation between inner and outer perimeters in Appendix 1. R_p =radius of curvature.

APPENDIX 2

To demonstrate the consistency, (3) will be derived using the same reasoning as used by WILLOUGHBY (1965). For notations, his work is referred to, in which curvilinear squares are introduced in the area between the electrodes, in a cross section. If this area is divided in m layers of n squares each, the capacitance is

$$C = \epsilon\epsilon_0 \frac{n}{m}$$

according to a known method for graphical evaluation of the capacitance. Now, for geometrical reasons, the square sizes increase outwards such that

$$\frac{dd_k}{dk} = ad_k,$$

where d_k is the side of a square in the layer number k and a a constant, i.e.

$$d_k = be^{ak},$$

the more exactly the finer the grid, i.e., the higher the values of n and m. Here b is another constant and $k = 0, \ldots, m$. At the surfaces we have $d_0 \approx p_1/n$ and $d_m \approx p_2/n$, where p_1 and p_2 are the perimeters of the inner and outer surfaces respectively. We therefore have the spacing

$$d_{pz} = \int_0^m d_k\,dk,$$

from which

$$\frac{n}{m} = \frac{p_2 - p_1}{d_{pz} \ln \frac{p_2}{p_1}}.$$

As $C = \epsilon\epsilon_0\, n/m$ one finds (3) when the notations of this paper are inserted (d instead of d_{pz}, p instead of p_1 and $p + 2\pi d$ instead of p_2—see Appendix 1). WILLOUGHBY (1965) uses the approximation $n/m \approx \sqrt{(p_1 p)_2}/d_{pz}$. To further stress the consistency, we will show how this may also be derived from the more exact relation above. We have for $x \approx 1$:

$$\ln x = \ln(1 + \sqrt{x} - 1) - \ln\left(1 + \frac{1}{\sqrt{x}} - 1\right) = \sqrt{x} - \frac{1}{\sqrt{x}} - \frac{1}{2}\left[x - \frac{1}{x} - 2\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right] + o\,[(\sqrt{x} - 1)^3] + o\left[\left(1 - \frac{1}{\sqrt{x}}\right)^3\right] = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\frac{1}{2}\left(4 - \sqrt{x} - \frac{1}{\sqrt{x}}\right) + o(\cdots) + o(\cdots),$$

if we put $\sqrt{x} = 1 + \epsilon$ with a small ϵ one easily shows

$$\ln x = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)[1 + o(\epsilon^3)] + o(\epsilon^3) + o\left(\frac{\epsilon^3}{\sqrt{x^3}}\right).$$

Here ordo symbols o have been used, $o(y)$ is a notation for a quantity which "approaches zero as y." Therefore, too a good accuracy

$$\ln x \approx \sqrt{x} - \frac{1}{\sqrt{x}}$$

if $x \approx 1$. The error is of the order $-\epsilon^3/3$ or $-\delta^3/24$ if we put $x = 1 + \delta$. This approximation for $\ln x$ leads to the expression for n/m used by WILLOUGHBY (1965), as $p_2 \approx p_1$ for a small d_{pz}, when inserted in the more exact relationship above.

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UNE ÉTUDE THÉORIQUE DE PLÉTHYSMOGRAPHIE CAPACITIVE

Sommaire—Une analyse théorique de pléthysmographie capacitive est présentée, qui est plus précise que celle d'une étude antérieure, due à une approximation mathématique plus exacte. Comme résultat, on montre que dans certaines circonstances, la pléthysmographie capacitive peut être assez imprécise, Les techniques mathématiques développées peuvent être utilisées pour chaque cas pratique, pour améliorer l'exactitude.

EINE THEORETISCHE UNTERSUCHUNG KAPAZITIVER PLETHYSMOGRAPHIE

Zusammenfassung—Es wird eine theoretische Analyse kapazitiver Plethysmographie gegeben, welche genauer, infolge exakterer mathematischer Annäherung, als die aus einer früheren Untersuchung ist. Demgemäss wird darauf hingewiesen, dass, unter bestimmten Bedingungen, kapazitive Plethysmographie recht ungenau sein kann. Die entwickelten mathematischen Verfahren können in jedem praktischen Falle zur Verbesserung der Genauigkeit verwandt werden.

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